

# Incomplete $h(x)$ -B-Tribonacci Polynomials

S. Arolkar<sup>1,\*</sup>, Y.S. Valaulikar<sup>2</sup>

<sup>1</sup>Department of Mathematics, D.M.'s College and Research Centre, Assagao-Bardez Goa, India, 403 507

<sup>2</sup>Department of Mathematics, Goa University, Taleigao Plateau, India, 403 206

\*Corresponding author: [suchita.golatkhar@yahoo.com](mailto:suchita.golatkhar@yahoo.com)

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**Abstract** In this paper we introduce incomplete  $h(x)$ -B-Tribonacci polynomials and obtain recurrence relations satisfied by a new class of polynomials.

**Keywords:** Incomplete Fibonacci Polynomials,  $h(x)$ -B-Tribonacci Polynomials and Incomplete  $h(x)$ -B-Tribonacci Polynomials

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## 1. Introduction

Fibonacci sequences and polynomials have many interesting properties and they provide wide opportunities to construct more fascinating properties of their own and their extensions see, for example, [3,4,8,10,11,12,13,14]. The B-Tribonacci sequence has been introduced in [1]. Incomplete Fibonacci and Lucas numbers and their properties have been studied in [5]. For further information about the incomplete generalized Fibonacci, Lucas and Tribonacci polynomials see, [6,7,9]. In this paper we now introduce incomplete  $h(x)$ -B-Tribonacci polynomials which are a natural extension of  $h(x)$ -Fibonacci polynomials earlier given by [6].

Let  $h(x)$  be a polynomial with real coefficients. The  $h(x)$ -B-Tribonacci polynomials  $({}^t\mathbf{B})_{h,n}(x)$ ,  $n \in \mathbb{N}$  are defined by

$$\begin{aligned}({}^t\mathbf{B})_{h,n+2}(x) &= h^2(x)({}^t\mathbf{B})_{h,n+1}(x) \\ &+ 2h({}^t\mathbf{B})_{h,n}(x) + ({}^t\mathbf{B})_{h,n-1}(x), \\ ({}^t\mathbf{B})_{h,0}(x) &= 0, ({}^t\mathbf{B})_{h,1}(x) = 0 \\ \text{and } ({}^t\mathbf{B})_{h,2}(x) &= 1,\end{aligned}\tag{1.1}$$

where the coefficients on the right hand side are the terms of binomial expansion of  $(h(x)+1)^2$  and  $({}^t\mathbf{B})_{h,n}(x)$  is the  $n^{\text{th}}$  polynomial. It is also given by

$$({}^t\mathbf{B})_{h,n}(x) = \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r}(x), \forall n \geq 2, \tag{1.2}$$

where  $(2n-4-2r)^r$  is  $(2n-4-2r)$  to the  $r$  falling factorial and  $\left\lfloor \frac{2n-4}{3} \right\rfloor$  is the largest integer not greater than  $\frac{2n-4}{3}$ .

Few polynomials of (1.1) are  $({}^t\mathbf{B})_{h,0}(x) = 0$ ,  $({}^t\mathbf{B})_{h,1}(x) = 0$ ,

$$({}^t\mathbf{B})_{h,2}(x) = 1, ({}^t\mathbf{B})_{h,3}(x) = h^2, ({}^t\mathbf{B})_{h,4} = h^4 + 2h$$

and

$$({}^t\mathbf{B})_{h,5}(x) = h^6 + 4h^3 + 1.$$

The generating function of (1.1) is given by

$$\sum_{n=0}^{\infty} ({}^t\mathbf{B})_{h,n} z^{n-2} = \frac{1}{1-z(h+z)^2}. \tag{1.3}$$

For further properties of (1.1), see [2]. We define the extension of incomplete  $h(x)$ -Fibonacci polynomials defined in [6] and call it as the incomplete  $h(x)$ -B-Tribonacci polynomials.

The incomplete  $h(x)$ -B-Tribonacci polynomials are defined by

$$\begin{aligned}({}^t\mathbf{B})_{h,n}^l(x) &= \sum_{r=0}^l \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r}(x), \\ 0 \leq l &\leq \left\lfloor \frac{2n-4}{3} \right\rfloor,\end{aligned}$$

and

$$n \geq 2. \tag{1.4}$$

Note that if  $l = \left\lfloor \frac{2n-4}{3} \right\rfloor$ , then

$$({}^t\mathbf{B})_{h,n}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} = ({}^t\mathbf{B})_{h,n}(x).$$

## 2. Some Recurrence Properties of the Polynomials $({}^t\mathbf{B})_{h,n}^l(x)$

In this section, we obtain some recurrence relations for  $({}^t\mathbf{B})_{h,n}^l(x)$  and some identities for a new class of

polynomials. For simplicity we use  $({}^tB)_{h,n}^l(x) = ({}^tB)_{h,n}^l$ ,

$$({}^tB)_{h,n}(x) = ({}^tB)_{h,n} \text{ and } h(x) = h.$$

**Proposition (2.1).** For  $n \geq 3$ , the recurrence relation of the incomplete  $h(x)$ -B- Tribonacci polynomials  $({}^tB)_{h,n}^l$  is

$$({}^tB)_{h,n+3}^{l+2} = h^2 ({}^tB)_{h,n+2}^{l+2} + 2h ({}^tB)_{h,n+1}^{l+1} + ({}^tB)_{h,n}^l, \quad (2.1)$$

$$0 \leq l \leq \left\lfloor \frac{2n-6}{3} \right\rfloor.$$

Using (1.4), Equ. (2.1) can be rewritten in terms of non-homogeneous recurrence relation as

$$({}^tB)_{h,n+3}^l = h^2 ({}^tB)_{h,n+2}^l + 2h ({}^tB)_{h,n+1}^l + ({}^tB)_{h,n}^l - \left[ \frac{(2n-4-2l)^l}{l!} h^{2n-4-3l} \right. \quad (2.2)$$

$$\left. - \left( 2 \frac{(2n-2-2l)^l}{l!} + \frac{(2n-2-2l)^{l-1}}{(l-1)!} \right) h^{2n-1-3l} \right].$$

**Proof.** This is proved by using (2.1) as follows.

$$\begin{aligned} & h^2 ({}^tB)_{h,n+2}^{l+2} + 2h ({}^tB)_{h,n+1}^{l+1} + ({}^tB)_{h,n}^l \\ &= h^2 \sum_{r=0}^{l+2} \frac{(2n-2r)^r}{r!} h^{2n-3r} \\ &+ 2h \sum_{r=0}^{l+1} \frac{(2n-2-2r)^r}{r!} h^{2n-2-3r} \\ &+ \sum_{r=0}^l \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &= \sum_{r=0}^{l+2} \frac{(2n-2r)^r}{r!} h^{2n+2-3r} \\ &+ 2 \sum_{r=0}^{l+1} \frac{(2n-2-2r)^r}{r!} h^{2n-1-3r} \\ &+ \sum_{r=0}^l \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &= h^{2n+2} + (2n-2) h^{2n-1} \\ &+ \sum_{r=2}^{l+2} \frac{(2n-2r)^r}{r!} h^{2n+2-3r} + 2h^{2n-1} \\ &+ 2 \sum_{r=1}^{l+1} \frac{(2n-2-2r)^r}{r!} h^{2n-1-3r} \\ &+ \sum_{r=0}^l \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &= h^{2n+2} + (2n) h^{2n-1} + \sum_{r=2}^{l+2} \frac{(2n-2r)^r}{r!} h^{2n+2-3r} \\ &+ 2 \sum_{r=2}^{l+2} \frac{(2n-2r)^{r-1}}{(r-1)!} h^{2n+2-3r} \\ &+ \sum_{r=2}^{l+2} \frac{(2n-2r)^{r-2}}{(r-2)!} h^{2n+2-3r} \end{aligned}$$

$$\begin{aligned} &= h^{2n+2} + (2n) h^{2n-1} \\ &+ \sum_{r=2}^{l+2} \left[ \frac{(2n-2r)^r}{r!} + 2 \frac{(2n-2r)^{r-1}}{(r-1)!} \right] h^{2n+2-3r} \\ &+ \sum_{r=2}^{l+2} \frac{(2n-2r)^{r-2}}{(r-2)!} h^{2n+2-3r} \\ &= \sum_{r=0}^{l+2} \frac{(2n+2-2r)^r}{r!} h^{2n+2-3r} \\ &= ({}^tB)_{h,n+3}^{l+2}. \end{aligned}$$

We now obtain some identities involving sums.

**Proposition (2.2).** For  $s \geq 1$ ,

$$\sum_{r=0}^{2s} \frac{(2s)^r}{r!} ({}^tB)_{h,n+r}^{l+r} h^r = ({}^tB)_{h,n+3s}^{l+2s}, \quad (2.3)$$

$$0 \leq l \leq \left\lfloor \frac{2n-2s-4}{3} \right\rfloor.$$

**Proof.** Follows from method of mathematical induction.

**Proposition (2.3).** For  $n \geq \left\lfloor \frac{3l+6}{2} \right\rfloor$  and  $s \geq 1$ , we have

$$\begin{aligned} & \sum_{r=0}^{s-1} \left( 2h^{2s-1-2r} ({}^tB)_{h,n+1+r}^{l+1} + h^{2s-2-2r} ({}^tB)_{h,n+r}^l \right) \quad (2.4) \\ &= ({}^tB)_{h,n+2+s}^{l+2} - h^{2s} ({}^tB)_{h,n+2}^{l+2}. \end{aligned}$$

**Proof.** The proof follows by using mathematical induction.

**Lemma (2.4).** For  $n \geq 2$ , we have

$$\begin{aligned} & \left[ \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \left( r \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \right) = \frac{2n-4}{3} ({}^tB)_{h,n} \right. \quad (2.5) \\ & \left. - \frac{h}{3} \sum_{r=0}^n \left( 2h ({}^tB)_{h,r+1} + 2 ({}^tB)_{h,r} \right) ({}^tB)_{h,n-r} \right. \end{aligned}$$

**Proof.** From (1.3), we have

$$\sum_{n=0}^{\infty} ({}^tB)_{h,n} z^{n-2} = \frac{1}{1-z(h+z)^2}.$$

Differentiating the above equation with respect to  $h$  we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\partial}{\partial h} \left( ({}^tB)_{h,n} \right) z^{n-2} = \frac{2hz}{[1-z(h+z)^2]^2} + \frac{2z^2}{[1-z(h+z)^2]^2} \\ &= 2hz \left[ \sum_{n=0}^{\infty} ({}^tB)_{h,n} z^{n-2} \right]^2 + 2z^2 \left[ \sum_{n=0}^{\infty} ({}^tB)_{h,n} z^{n-2} \right]^2 \\ &= 2hz^{-3} \left[ \sum_{n=0}^{\infty} ({}^tB)_{h,n} z^n \right]^2 + 2z^{-2} \left[ \sum_{n=0}^{\infty} ({}^tB)_{h,n} z^n \right]^2 \\ &= 2h \sum_{n=0}^{\infty} \sum_{r=0}^n \left( ({}^tB)_{h,r} ({}^tB)_{h,n-r} \right) z^{n-3} \\ &+ 2 \sum_{n=0}^{\infty} \sum_{r=0}^n \left( ({}^tB)_{h,r} ({}^tB)_{h,n-r} \right) z^{n-2} \end{aligned}$$

Comparing both sides the coefficient of  $z^{n-2}$ , we have

$$\frac{\partial}{\partial h} \left( ({}^t B)_{h,n} \right) = 2h \sum_{r=0}^{n+1} ({}^t B)_{h,r} ({}^t B)_{h,n+1-r} + 2 \sum_{r=0}^n ({}^t B)_{h,r} ({}^t B)_{h,n-r}$$

That is

$$\frac{\partial}{\partial h} \left( ({}^t B)_{h,n} \right) = \sum_{r=0}^n \left( 2h ({}^t B)_{h,r+1} + 2({}^t B)_{h,r} \right) ({}^t B)_{h,n-r}. \quad (2.6)$$

Also differentiating (1.2) on both sides with respect to  $h$ , we get

$$\begin{aligned} \frac{\partial}{\partial h} \left( ({}^t B)_{h,n} \right) &= \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^r (2n-4-3r)}{r!} h^{2n-5-3r} \\ &= \frac{2n-4}{h} \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &\quad - \frac{3}{h} r \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \end{aligned}$$

Thus, we have

$$\begin{aligned} &r \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &= \frac{2n-4}{3} ({}^t B)_{h,n} - \frac{h}{3} \frac{\partial}{\partial h} \left( ({}^t B)_{h,n} \right) \\ &= \frac{2n-4}{3} ({}^t B)_{h,n} - \frac{h}{3} \sum_{r=0}^n \left( 2h ({}^t B)_{h,r+1} + 2({}^t B)_{h,r} \right) ({}^t B)_{h,n-r}, \end{aligned}$$

This completes the proof.

**Proposition (2.5).** For  $n \geq 2$ , we have

$$\begin{aligned} \sum_{l=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} ({}^t B)_{h,n}^l &= \left( \left\lfloor \frac{2n-4}{3} \right\rfloor - \frac{2n-7}{3} \right) ({}^t B)_{h,n} \\ &\quad + \frac{h}{3} \sum_{r=0}^n \left( 2h ({}^t B)_{h,r+1} + 2({}^t B)_{h,r} \right) ({}^t B)_{h,n-r} \end{aligned}$$

**Proof.** From (1.4), the sum,

$$\begin{aligned} \sum_{l=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} ({}^t B)_{h,n}^l &= ({}^t B)_{h,n}^0 + ({}^t B)_{h,n}^1 + \dots \\ &\quad + ({}^t B)_{h,n}^k + \dots + ({}^t B)_{h,n}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \\ &= \sum_{r=0}^0 \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &\quad + \sum_{r=0}^1 \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \end{aligned}$$

$$\begin{aligned} &+ \dots + \sum_{r=0}^k \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} + \dots \\ &+ \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &= \frac{(2n-4)^0}{0!} h^{2n-4} + \left[ \frac{(2n-4)^0}{0!} h^{2n-4} + \frac{(2n-6)^1}{1!} h^{2n-7} \right] \\ &+ \left[ \frac{(2n-4)^0}{0!} h^{2n-4} + \frac{(2n-6)^1}{1!} h^{2n-7} + \frac{(2n-8)^2}{2!} h^{2n-10} \right] + \dots \\ &+ \left[ \frac{(2n-4)^0}{0!} h^{2n-4} + \frac{(2n-6)^1}{1!} h^{2n-7} + \dots + \frac{(2n-4-2k)^k}{k!} h^{2n-4-3k} \right] + \dots \\ &+ \left[ \frac{(2n-4)^0}{0!} h^{2n-4} + \frac{(2n-6)^1}{1!} h^{2n-7} + \dots + \frac{\left( 2n-4-2 \left\lfloor \frac{2n-4}{3} \right\rfloor \right)^{\left\lfloor \frac{2n-4}{3} \right\rfloor}}{\left( \left\lfloor \frac{2n-4}{3} \right\rfloor \right)!} h^{2n-4-3 \left\lfloor \frac{2n-4}{3} \right\rfloor} \right] \\ &= \left( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 \right) \frac{(2n-4)^0}{0!} h^{2n-4} \\ &+ \left( \left\lfloor \frac{2n-4}{3} \right\rfloor \right) \frac{(2n-6)^1}{1!} h^{2n-7} \\ &+ \left( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 - r \right) \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &+ \frac{\left( 2n-4-2 \left\lfloor \frac{2n-4}{3} \right\rfloor \right)^{\left\lfloor \frac{2n-4}{3} \right\rfloor}}{\left( \left\lfloor \frac{2n-4}{3} \right\rfloor \right)!} h^{2n-4-3 \left\lfloor \frac{2n-4}{3} \right\rfloor} \\ &= \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \left( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 - r \right) \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &= \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \left( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 \right) \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &\quad - r \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &= \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \left( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 \right) \frac{(2n-4-2r)^r}{r!} h^{2n-4-3r} \\ &\quad - \frac{2n-4}{3} ({}^t B)_{h,n} \end{aligned}$$

$$+ \frac{h}{3} \sum_{r=0}^n \left( 2h \binom{t}{h,r+1} + 2 \binom{t}{h,r} \right) \binom{t}{h,n-r}.$$

In view of (2.5), we complete the proof with the following application.

$$\begin{aligned} \sum_{l=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \binom{t}{h,n}^l &= \left( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 \right) \binom{t}{h,n} - \frac{2n-4}{3} \binom{t}{h,n} \\ &\quad + \frac{h}{3} \sum_{r=0}^n \left( 2h \binom{t}{h,r+1} + 2 \binom{t}{h,r} \right) \binom{t}{h,n-r} \\ &= \left( \left\lfloor \frac{2n-4}{3} \right\rfloor - \frac{2n-7}{3} \right) \binom{t}{h,n} \\ &\quad + \frac{h}{3} \sum_{r=0}^n \left( 2h \binom{t}{h,r+1} + 2 \binom{t}{h,r} \right) \binom{t}{h,n-r}. \end{aligned}$$

### 3. Conclusion

In this paper, we have defined incomplete  $h(x)$ -B-Tribonacci polynomials and obtained some identities related to these polynomials.

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